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On the Distribution of Numbers

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This paper examines the distribution of the mantissas of floating point numbers and shows how the arithmetic operations of a computer transform various distributions toward the limiting distribution

$$r(x) = \frac{1}{x \ln b} \qquad (1/b \leqq x \leqq 1)$$

(where b *is the base of the number system). The paper also gives a number of applications to hardware, software, and general computing which show that this distribution is not merely an amusing curiosity. A brief examination of the distribution of exponents is included.*

I. INTRODUCTION

The main purpose of this paper is to examine, from the computing machine's point of view, the well-known (to comparatively few people) unequal distribution of the "mantissas" of "naturally occurring" sets of numbers. The observed probability density distributions are often close to the reciprocal density distribution

$$r(t) = \frac{1}{t \ln b} \qquad (1/b \leqq t \leqq 1), \tag{1}$$

where b is the number base (usually 2, 8, 10, or 16). The corresponding cumulative probability distribution is

$$R(t) = \int_{1/b}^{t} r(x)\, dx = \int_{1/b}^{t} \frac{dx}{x \ln b}$$

$$= \frac{\ln t + \ln b}{\ln b} \tag{2}$$

where, of course,

$$R(1/b) = 0 \quad \text{and} \quad R(1) = 1.$$

From the cumulative distribution, it follows that the probability of observing the leading digit N of a number that is drawn at random from $r(t)$ is

$$R(N + 1) - R(N) = \frac{\ln (N + 1) - \ln (N)}{\ln b}, \tag{3}$$

and this is usually what is measured in experiments.

A typical experiment is that of tabulating the number of physical constants in a table having a given leading digit (see Table I and Ref. 1, p. 7). The result looks reasonable. Many other examples of observing the reciprocal distribution have been reported. For references see Refs. 2 and 3.

The reciprocal distribution has been explained in many ways. One popular but not immediately obvious explanation for the distribution of physical constants is as follows. Consider the distribution of the leading

Table I—The Distribution of the Leading Digits of 50 Physical Constants

Leading digit N	Number of cases observed	Expected number eq. (3)	Difference
1	16	15	1
2	11	9	2
3	2	6	−4
4	5	5	0
5	6	4	2
6	4	3	1
7	2	3	−1
8	1	3	−2
9	3	2	1
	50	50	

digits of the set of *all* the physical constants that might occur. If the units of measurement were to be changed then the corresponding leading digit of any particular physical constant would probably change, but it is difficult to believe that the distribution itself would change significantly. To believe so seems to indicate a belief that either the present units of measurement or else the new set have some intimate connection with the real world. An alternative, and more elegant, explanation is given by Roger Pinkham in his classic paper (Ref. 2). The explanation given in the present paper is based on how the computer transforms distributions during arithmetic operations. In particular the paper shows how, from any reasonable distributions, repeated multiplications and/or divisions rapidly move the distributions toward the reciprocal distribution. The effect for addition and subtraction is somewhat different. The paper also shows the persistance of the reciprocal distribution once it is attained.

Since floating point numbers are the basis of most of numerical analysis one may well ask why this obvious and experimentally well-verified distribution is so often ignored. Is it because it appears to contradict the usually accepted model of the number system in which numbers correspond to points on a homogeneous straight line? Not only are the floating point numbers not uniformly spaced in a computer (the difference between the two largest possible numbers is very large, while the distance between the two smallest positive number is very small, and zero is relatively isolated), but the reciprocal distribution shows that even in intervals in which the numbers are equally spaced they are not equally likely to occur.

Thus in analogy with non-Euclidean geometry this paper proposes an alternative to the conventional identification of numbers with points on a homogeneous straight line. Instead of adopting a measure for sets that is invariant under translation

$$x' = x + k,$$

we often prefer a measure that is invariant under scaling, namely

$$x' = kx \qquad (k \neq 0).$$

The reciprocal distribution is of practical as well as theoretical interest as we shall show in Section VII. In view of these examples, it is hoped that by adopting the machine's point of view with respect to how numbers are transformed by arithmetical operations, the computer scientists will become more aware of the importance of this distribution in many situations including numerical analysis.

II. THE MODEL

The floating point numbers in a computing machine form a discrete, finite set. As is true in so many applications of mathematics to practical problems, we shall approximate a discrete distribution by a continuous one of sufficient smoothness. Anyone familiar with the upper and lower Riemann Integral sums can appreciate the degree of approximation being made (provided common sense is used in choosing the values of the curve between the given points). In the limit of the Riemann sum all the $|\Delta x_i|$ become less than any given $\epsilon > 0$; we of course need to stop at the granularity of the number system used, typically 10^{-8} or smaller.

In principle, it is possible to carry this error estimate throughout all the subsequent steps of the mathematics to see how much the mathematics errs from reality; but it is customary to recognize that a little intuition will suffice to convince the user that the error will be much less than the accuracy of the experiments that the theory is designed to account for. Thus we have no need to get excited about such things as the Banach measure of a set (Ref. 4); we do not intend in this paper to let the mathematics obscure what is going on. The fact that computers are finite and operate at a finite speed for a finite length of time spares us from taking seriously all the confusions that can arise in mathematics when dealing with the infinite.

III. THE BASIC FORMULAS

In this section we derive the basic formulas which describe how distributions are combined and transformed by the four arithmetic operations of a computer. Let $f(x)$ be the density distribution of the factor x, $g(y)$ be the density distribution of the factor y, and $h(z)$ be the density distribution of the result z of the arithmetic operation. Further, let $F(x)$, $G(y)$, and $H(z)$ be the corresponding cumulative distributions.

For both multiplication and division, the mantissas are directly combined and the exponents do not enter into the formation of the distribution of the result of the operation. Thus, it is sufficient in these cases to consider the distributions for $(1/b \leqq x, y \leqq 1)$.

For multiplication, an examination of Fig. 1 shows that when the product falls in the shaded regions then the mantissa of the product is in the interval $(1/b, z)$. Thus the cumulative distribution $H(z)$ is given by

$$H(z) = \int_{1/b}^{z} \int_{1/b}^{z/bx} f(x)g(y)\,dy\,dx + \int_{1/b}^{z} \int_{1/bx}^{1} f(x)g(y)\,dy\,dx + \int_{z}^{1} \int_{1/bx}^{z/x} f(x)g(y)\,dy\,dx$$

$$= \int_{1/b}^{z} f(x)[G(z/bx) - G(1/b) + G(1) - G(1/bx)]\, dx$$

$$+ \int_{z}^{1} f(x)[G(z/x) - G(1/bx)]\, dx.$$

Differentiating with respect to z to get the density distribution we have

$$h(z) = f(z)[G(1/b) - G(1/b) + G(1) - G(1/bz) - G(1) + G(1/bz)]$$

$$+ \int_{1/b}^{z} f(x)g(z/bx)(1/bx)\, dx + \int_{z}^{1} f(x)g(z/x)(1/x)\, dx$$

$$= \frac{1}{b}\int_{1/b}^{z} \frac{f(x)}{x}\, g(z/bx)\, dx + \int_{z}^{1} \frac{f(x)}{x}\, g(z/x)\, dx. \tag{4}$$

Similarly for division. The shaded region of Fig. 2 shows where the quotient x/y is less than z; thus the cumulative distribution for the quotient is

$$H(z) = \int_{1/b}^{z}\int_{1/b}^{x} f(x)g(y)\, dy\, dx + \int_{1/b}^{z}\int_{x/z}^{1} f(x)g(y)\, dy\, dx$$

$$+ \int_{z}^{1}\int_{x/bz}^{x} f(x)g(y)\, dy\, dx$$

$$= \int_{1/b}^{z} f(x)[G(x) - G(1/b) + G(1) - G(x/z)]\, dx$$

$$+ \int_{z}^{1} f(x)[G(x) - G(x/bz)]\, dx.$$

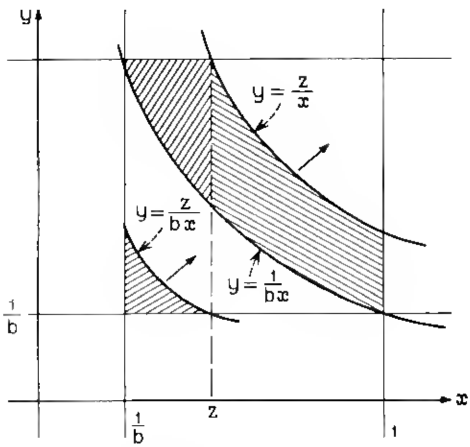

Fig. 1—The cumulative probability distribution for the product $z = xy$.

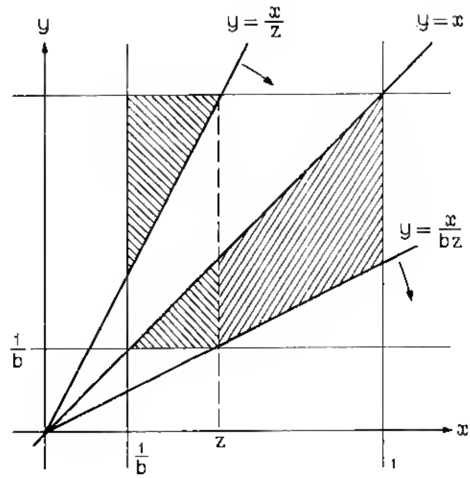

Fig. 2—The cumulative probability distribution for the quotient $z = x/y$.

Again differentiating with respect to z to get the density distribution we have

$$\begin{aligned} h(z) &= f(z)[G(z) - G(1/b) + G(1) - G(1) - G(z) + G(1/b)] \\ &\quad + \int_{1/b}^{z} f(x)[-g(x/z)(-x/z^2)]\,dx + \int_{z}^{1} f(x)[-g(x/bz)(-x/bz^2)]\,dx \\ &= \frac{1}{z^2}\int_{1/b}^{z} xf(x)g(x/z)\,dx + \frac{1}{bz^2}\int_{z}^{1} xf(x)g(x/bz)\,dx. \end{aligned} \tag{5}$$

For both addition and subtraction the difference in the exponents of the two numbers x and y is used to shift one mantissa with respect to the other *before* they are combined. For addition, we may suppose that one of the numbers, say x, lies in the range $z/2 \leqq x \leqq z$. The other term, y, therefore lies in the range $z/2 \geqq y \geqq z \cdot b^{-k}$, where k is the number of digits in the mantissa and we set $b^{-k} = \epsilon$. Thus the density distribution of the sum is

$$h(z) = \int_{z/2}^{z(1-\epsilon)} f(x)g(z - x)\,dx. \tag{6}$$

For subtraction we suppose, without loss of generality, that $x \geqq y > 0$, and

$$z = x - y$$

with $z \leqq x \leqq z/\epsilon$. Then the density distribution is given by

$$h(z) = \int_z^{z/\epsilon} x)g(z + x)\, dx. \tag{7}$$

We have now derived the basic relations for the density distributions that arise from combining two numbers from arbitrary distributions according to the four arithmetic operations of a computer.

IV. THE PERSISTENCE OF THE RECIPROCAL DISTRIBUTION

In this section, we first show for both multiplication and division that if one of the factors x or y comes from the reciprocal distribution, and regardless of the distribution of the other factor, then $h(z)$ is the reciprocal distribution. In particular, if a number is chosen from the reciprocal distribution, then its reciprocal is also from the reciprocal distribution. For addition and subtraction we show somewhat less.

For the product set

$$g(y) = \frac{1}{y \ln b} \tag{8}$$

in equation (4). We get for any distribution $f(x)$

$$\begin{aligned} h(z) &= \frac{1}{b}\int_{1/b}^{z} \frac{f(x)}{x}\cdot\frac{bx}{z \ln b}\, dx + \int_z^1 \frac{f(x)}{x}\,\frac{x}{z \ln b}\, dx \\ &= \frac{1}{z \ln b}\left[\int_{1/b}^{z} f(x)\, dx + \int_z^1 f(x)\, dx\right] = \frac{1}{z \ln b}. \end{aligned} \tag{9}$$

Obviously since $z = xy$, the same applies if we assume that $f(x)$ is the reciprocal distribution.

For the quotient, again assume equation (8) and put it in equation (5).

$$\begin{aligned} h(z) &= \frac{1}{z^2}\int_{1/b}^{z} xf(x)\,\frac{z}{x \ln b}\, dx + \frac{1}{bz^2}\int_z^1 xf(x)\,\frac{bz}{x \ln b}\, dx \\ &= \frac{1}{z \ln b}\left\{\int_{1/b}^{z} f(x)\, dx + \int_z^1 f(x)\, dx\right\} = \frac{1}{z \ln b}. \end{aligned} \tag{10}$$

In the special case of $f(x)$ being the "spike distribution" with all of its probability at $x = 1$ we see that the reciprocal of a variable having the reciprocal distribution has the reciprocal distribution. The case of x having the reciprocal distribution and producing the reciprocal distribution, regardless of the distribution of the denominator, is covered by the product form, or can be worked out directly if desired.

Thus, if in a long sequence of multiplications and divisions at least one

factor bas tbe reciprocal distribution, then regardless of how the distributions of the otber factors are chosen the result is still tbe reciprocal distribution; the reciprocal distribution persists under multiplication and division and cannot be broken by any cboices for tbe otber factors.

For addition let x come from the reciprocal distribution for some range witb normalization factor N_1 , and y also come from a reciprocal distribution with its corresponding range and normalization factor N_2 . Tben writing $\epsilon = b^{-k}$

$$\begin{aligned} h(z) \quad & \int_{z/2}^{z(1-\epsilon)} \frac{N_1}{x} \cdot \frac{N_2}{z - x} \, dx \\ &= N_1 N_2 \int_{z/2}^{z(1-\epsilon)} \frac{1}{z} \left[\frac{1}{x} + \frac{1}{z - x} \right] dx \\ &= \frac{N_1 N_2}{z} \ln \left[\frac{x}{z - x} \right] \Bigg|_{z/2}^{z(1-\epsilon)} \\ &= \frac{N_3}{z} \end{aligned} \tag{11}$$

where N_3 is some constant.

Similarly for subtraction (different N_i)

$$\begin{aligned} h(z) &= \int_{z}^{z/\epsilon} \frac{N_1}{x} \cdot \frac{N_2}{z + x} \, dx \\ &= N_1 N_2 \int_{z}^{z/\epsilon} \frac{1}{z} \left[\frac{1}{x} - \frac{1}{z + x} \right] dx \\ &= \frac{N_1 N_2}{z} \ln \left[\frac{x}{z + x} \right] \Bigg|_{z}^{z/\epsilon} \\ &= \frac{N_3}{z}. \end{aligned} \tag{12}$$

It should be noted, however, that in the last two cases the assumption of the reciprocal distribution for such great ranges is suspicious to say the least, since we know from experience that all exponents are not equally likely. That tbe reciprocal distribution over a large range implies the equally likely distribution of tbe relevant exponents can be seen by examining the base 16 number system in exponents, but wbere tbe mantissas are in binary. Tbus tbe mantissas can have one of the forms:

$$0.1xxx \cdots$$

$$0.01xx \cdots$$

$$0.001x \cdots$$

$$0.0001 \cdots .$$

If we assume

$$p(x) = \frac{1}{x \ln 16} \qquad (\tfrac{1}{16} \leqq x \leqq 1),$$

what are the probabilities of each of the four forms? For the first one

$$\int_{\frac{1}{2}}^{1} \frac{1}{x \ln 16}\, dx = \frac{1}{4 \ln 2}\,[\ln 1 - \ln \tfrac{1}{2}]$$
$$= \tfrac{1}{4}.$$

Similarly, each of the others is $\frac{1}{4}$. This result is quite different from that of the flat distribution (see Table II).

V. THE APPROACH TO THE RECIPROCAL DISTRIBUTION

Having shown that once it arises the reciprocal distribution persists for multiplication and division, we need to show how it can arise. For this we need a measure of how far a distribution $h(z)$ is from the reciprocal distribution $r(z)$. It is obvious that

$$\int_{1/b}^{1} [h(z) - r(z)]\, dz = 0 \tag{13}$$

for any $h(z)$ and this does not provide a useful measure of distance. We shall define the distance of $h(z)$ from the reciprocal distribution $r(z)$ by

$$\max_{1/b \leqq z \leqq 1} \left| \frac{h(z) - r(z)}{r(z)} \right| \equiv D\{h(z)\} = D\{h\}, \tag{14}$$

which measures the maximum of the difference *relative* to the reciprocal distribution (it is natural to use the relative error when dealing with floating point numbers).

Table II—Probability of Observing Mantissas With Leading Zeros in Base 16 Numbers When Written in Base 2

Form	Range	Binary Exponent	Probabilities	
			Flat	Reciprocal
0.0001....	$1/16 \leqq x \leqq 1/8$	−3	1/15	1/4
0.001x....	$1/8 \leqq x \leqq 1/4$	−2	2/15	1/4
0.01xx....	$1/4 \leqq x \leqq 1/2$	−1	4/15	1/4
0.1xxx....	$1/2 \leqq x \leqq 1$	0	8/15	1/4

We showed in equation (9) that for a product,

$$r(z) = \frac{1}{b}\int_{1/b}^{z} \frac{f(x)}{x} r(z/bx)\, dx + \int_{z}^{1} \frac{f(x)}{x} r(z/x)\, dx.$$

Subtracting this from equation (4) and dividing by $r(z)$ we have

$$\frac{h(z) - r(z)}{r(z)} = \frac{1}{b}\int_{1/b}^{z} \frac{f(x)}{x}\left[\frac{g(z/bx) - r(z/bx)}{r(z)}\right] dx + \int_{z}^{1} \frac{f(x)}{x}\left[\frac{g(z/x) - r(z/x)}{r(z)}\right] dx.$$

But

$$bxr(z) = \frac{bx}{z \ln b} = r(z/bx)$$

$$xr(z) = \frac{x}{z \ln b} = r(z/x),$$

and we have

$$\frac{h(z) - r(z)}{r(z)} = \int_{1/b}^{z} f(x)\left[\frac{g(z/bx) - r(z/bx)}{r(z/bx)}\right] dx + \int_{z}^{1} f(x)\left[\frac{g(z/x) - r(z/x)}{r(z/x)}\right] dx. \tag{15}$$

Since $f(x) \geqq 0$ for $(1/b \leqq x \leqq 1)$,

$$\left|\frac{h(z) - r(z)}{r(z)}\right| \leqq \int_{1/b}^{z} f(x)\, D\{g\}\, dx + \int_{z}^{1} f(x)\, D\{g\}\, dx \leqq D\{g\}$$

for all z. From this it follows that

$$D\{h\} \leqq D\{g\} \tag{16}$$

regardless of the choice of $f(x)$.

We note that the equality would hold if $f(x)$ were a single spike at $x = 1$, say, but that in view of equation (13), we generally expect a great deal of cancellation in the square brackets of equation (15) as it is integrated over the range.

It is easy to examine the rapidity of the approach in the case of all the factors coming from the flat distribution

$$p(x) = \frac{1}{1 - 1/b} = \frac{b}{b - 1}.$$

Equation (14) gives for two factors

$$h(z) = \frac{1}{b}\left(\frac{b}{b-1}\right)^2 \int_{1/b}^{z} \frac{dx}{x} + \left(\frac{b}{b-1}\right)^2 \int_{z}^{1} \frac{dx}{x}$$

$$= \frac{b}{(b-1)^2}\{\ln b - (b-1)\ln z\}.$$

In the base $b = 10$, this is

$$h(z) = \tfrac{10}{81}\{\ln 10 - 9 \ln z\}, \tag{17}$$

which (for the proper range) is given by Ref. 5 (p. 37). The distance of the flat distribution is

$$\max_{1/10 \leqq x \leqq 1} \left| z\,\frac{10 \ln 10}{9} - 1 \right| = \frac{10 \ln 10}{9} - 1 = 1.558...$$

while the distance of equation (17) is equal to $0.3454\cdots$. See Table III for further results.

Similarly for division using equations (10) and (5), we have

$$\frac{h(z) - r(z)}{r(z)} = \frac{1}{z^2}\int_{1/b}^{z} xf(x)\left[\frac{g(x/z) - r(x/z)}{r(z)}\right] dx$$

$$+ \frac{1}{bz^2}\int_{z}^{1} xf(x)\left[\frac{g(x/bz) - r(x/bz)}{r(z)}\right] dx.$$

But

$$z^2\,\frac{r(z)}{x} = r(x/z)$$

$$bz^2\,\frac{r(z)}{x} = r(x/bz),$$

and we have

$$\left|\frac{h(z) - r(z)}{r(z)}\right| \leqq D\{g\}\left\{\int_{1/b}^{z} f(x)\,dx + \int_{z}^{1} f(x)\,dx\right\}$$

TABLE III—THE DISTANCE OF A CONTINUED PRODUCT AS A FUNCTION OF THE NUMBER OF FACTORS SELECTED FROM A FLAT DISTRIBUTION

Number of Factors	Distance
1	1.558
2	0.3454
3	0.0980
4	0.0289

or

$$D\{h\} \leqq D\{g\}.$$

In the case of flat distributions

$$h(z) = \frac{1}{2(b-1)}\left[b + \frac{1}{z^2}\right]$$

which for the base 10 is (see Ref. 5, p. 37)

$$h(z) = \frac{1}{18}\left[10 + \frac{1}{z^2}\right]$$

and has a distance of 0.4071 $\cdots$.

For addition we select $g(y)$ as a reciprocal distribution (with suitable normalization factor N), subtract the corresponding equations and divide by $r(z)$ to get

$$\frac{h(z) - r(z)}{r(z)} = \int_{z/2}^{z(1-\epsilon)} \left[f(x)\frac{N_2}{z-x} - \frac{N_1}{x}\frac{N_2}{z-x}\right]\frac{dx}{r(z)}$$

$$= \int_{z/2}^{z(1-\epsilon)} \left[\frac{f(x) - \dfrac{N_1}{x}}{r(x)}\right] \cdot \left[\frac{N_2}{z-x}\frac{r(x)}{r(z)}\right] dx.$$

But by the mean value theorem for integrals

$$\frac{h(z) - r(z)}{r(z)} = \left[\frac{f(\theta) - \dfrac{N_1}{\theta}}{r(\theta)}\right] \int_{z/2}^{z(1-\epsilon)} \frac{N_2}{z-x} \cdot \frac{r(x)}{r(z)}\, dx,$$

where $z/2 \leqq \theta \leqq z(1-\epsilon)$. The integral has been shown in equation (11) to be exactly 1. Hence

$$D\{h(z)\} \leqq D\{f(x)\}.$$

A similar derivation works for subtraction.

In view of the dubious assumption of having the reciprocal distribution over a very large range we need to examine more carefully the behavior of the mantissas of sums of numbers selected from some distribution. Let us imagine a Monte Carlo experiment. We select numbers from the range $(0 < a \leqq x \leqq b)$ having the probability density distribution $p(x)$ with mean μ and variance σ^2. Divide the range into n equal intervals

$$(a, a+h), \qquad (a+h, a+2h), \cdots, [a+(n-1)h, b],$$

where $h = (b - a)/n$. By counting how many numbers fall in each interval we get estimates of $p(x)$.

Let us add 2^m numbers of this set of numbers. The range for the sum is

$$(2^m a,\ 2^m b),$$

the mean $\mu_1 = 2^m \mu$ and $\sigma_1^2 = 2^k \sigma^2$. But the central limit theorem says that the distribution of the sum approaches a normal distribution about the mean with half width σ_1. Suppose, for convenience, that μ fell in the middle of an interval. Then as m increases and we count the number of cases of mantissas in each interval (note that the m in the term 2^m appears in the exponent only) we will find more and more of them will fall in the interval containing μ (which has the same mantissa as μ_1); the distribution approaches a spike! This does not contradict the central limit theorem; it merely says that if $\mu \neq 0$ ($\mu = 0$ is the exceptional case), the distribution contracts as seen from the point of view of floating point numbers. In loose words, standing at the origin and viewing the rapidly receding mean μ_1, the width of the distribution σ_1 seems to get narrower as compared to the sum—the sum recedes as 2^m, the half width changes as $2^{m/2}$.

VI. RANGE OF EXPONENTS

It is now clear that in order to examine carefully the effect of addition (and subtraction) on the reciprocal distribution, it is necessary to know the distribution of the exponents of the numbers to be combined. Unfortunately at this time about the only model we have is as follows. Assume a distribution of exponents. Under multiplication and division the exponents are added and subtracted (with, due to carries an extra 1 occasionally added, or subtracted) and by the central limit theorem we can expect: (*i*) that the distribution of the exponents will approach a normal distribution (assuming that overflow and underflow do not happen first) and (*ii*) that this distribution will gradually spread out proportional to the square root of the number of operations. Thus, it appears that in practice the distribution of exponents is probably not stationary. Addition tends to eliminate the smaller exponents, while subtraction tends to increase them.

Experience in numerical analysis shows that the range of the output numbers is usually much greater than the range of the input numbers, enough so to make one suspect that the variance increases as indicated in the above model.

As one thinks carefully about the matter of addition and subtraction it seems reasonable to believe that they will not greatly perturb the

reciprocal distribution; and the experimental data from "naturally occurring numbers", which must have included some additions and subtractions, seem to bear out this belief.

The feeling that under repeated additions and subtractions the central limit theorem applies to numbers (which is true), and therefore contradicts the reciprocal distribution of the mantissas, is typical of the "fixed point arithmetic" viewpoint of numbers—we are representing the sums and differences as floating point numbers, and it is the distribution of these mantissas and their possible approach to the reciprocal distribution that is of relevance here.

VII. APPLICATIONS OF THE RECIPROCAL DISTRIBUTION

Besides accounting for the experimentally found distributions, the reciprocal distribution is relevant to many optimization situations.

As a first example,[6] consider the problem of placing the decimal (binary) point in the number representation system in order to minimize the number of normalization shifts after the computation of a product. (It was probably the minimization of normalizing shifts that caused IBM to adopt the base 16 in the system 360). If the point is placed before the first digit, then products of the form

$$\begin{array}{r} 0.xxx\ldots \\ \underline{0.xxx\ldots} \\ 0.0xx\ldots \end{array}$$

will require a shift to normalize the result; while if it is placed after the first digit, then products like

$$\begin{array}{r} x.x\ldots \\ \underline{x.x\ldots} \\ xx.x\ldots \end{array}$$

will require a shift. Clearly these two cases have complementary probabilities. For the reciprocal distributions the probability p of

$$xy \leq 1/b$$

is

$$\begin{aligned} p &= \int_{1/b}^{1} \int_{1/b}^{1/bx} \frac{1}{x \ln b} \frac{1}{y \ln b} \, dy \, dx \\ &= \int_{1/b}^{1} \frac{1}{\ln^2 b} \left(\frac{\ln 1/bx - \ln 1/b}{x} \right) dx = \int_{1/b}^{1} \frac{1}{\ln^2 b} \left(-\frac{\ln x}{x} \right) dx \\ &= \frac{1}{\ln^2 b} \left\{ -\frac{\ln^2 x}{2} \right\} \Bigg|_{1/b}^{1} = \frac{1}{2}. \end{aligned}$$

But for a flat distribution,

$$p = \left(\frac{b}{b-1}\right)^2 \int_{1/b}^{1} \int_{1/b}^{1/bx} dy\, dx = \left(\frac{b}{b-1}\right)^2 \int_{1/b}^{1} \left(\frac{1}{bx} - \frac{1}{b}\right) dx$$

$$= \left(\frac{b}{b-1}\right)^2 \frac{1}{b} \left[\ln b - \left(1 - \frac{1}{b}\right)\right]$$

$$= \frac{b \ln b - (b-1)}{(b-1)^2}.$$

For $b = 2$ this is

$$p = 2 \ln 2 - 1 \cong 0.38.$$

As a second application, consider the estimation of the effect of the representation error of numbers in base 2 and base 16. In Ref. 7 McKeeman reports that the maximum relative representation error (MRRE) and the average relative representation error (ARRE) are as shown in Table IV, where the average is over the reciprocal distribution.

A third example is the application to roundoff propagation. If x_1 has an error ϵ_1 and x_2 has error ϵ_2, then in the product

$$\begin{array}{r} x_1 + \epsilon_1 \\ x_2 + \epsilon_2 \\ \hline x_1x_2 + x_1\epsilon_2 + x_2\epsilon_1 + \epsilon_1\epsilon_2 \end{array}$$

it is the leading digits that control the estimate of the propagated error. For the reciprocal distribution the mean is

$$\bar{x} = \int_{1/b}^{1} \frac{x}{x \ln b}\, dx = \frac{1 - 1/b}{\ln b} = \frac{b-1}{b \ln b}.$$

For base 2, this is

$$\bar{x} = \frac{1}{2 \ln 2} \cong 0.72134.$$

TABLE IV—MAXIMUM RELATIVE REPRESENTATION ERROR AND AVERAGE RELATIVE REPRESENTATION ERROR

	MRRE	ARRE
binary	$1/2 \times 2^{-37}$	0.18×2^{-37}
octal	2^{-37}	0.21×10^{-37}
hexadecimal	2^{-37}	0.17×2^{-37}

The second moment about the mean is

$$M_2 = \frac{1}{\ln b} \int_{1/b}^{1} \frac{(x - \bar{x})^2}{x} \, dx = \frac{b - 1}{b^2 \ln b} \left\{ \frac{b + 1}{2} - \frac{b - 1}{\ln b} \right\}$$

which for $b = 2$ is

$$M_2 = \frac{1}{4 \ln 2} \left(\frac{3}{2} - \frac{1}{\ln 2} \right) \cong 0.020674.$$

For the flat distribution, $\bar{x} = 0.75$ and $M_2 = 0.020833$.

Thus we see that the effect of the reciprocal distribution on the average roundoff propagation is surprisingly small.

Another example in which the reciprocal distribution must be considered is that of producing "random" floating point mantissas. To generate these mantissas we use the earlier result that a long sequence of multiplications of numbers from a flat distribution will approximate a reciprocal distribution. Thus random mantissas can be generated by

$$Y_n = Y_{n-1} \cdot r_n \qquad \text{(shifted)}$$

where r_n is from the usual (flat) random number generator and "shifted" means after each product the leading zeros are shifted off. How well does this work? Experimental verification* is given by 8192 trials. Counting the number of mantissas falling in each of N catagories (see Table V).

The last two columns of Table V give the sign changes observed in the difference between the observed and theoretical reciprocal distribution. The expected number of sign changes might be expected to be $(N - 1)/2$, but since for $N = 2$ it is clear that one sign change will occur (because the mean of the residuals is zero) we have used $N/2$ as the expected number. The chi-square test shows that the two distributions are close; the sign change test shows that the residuals are not systematically distributed. From these tests, we see that the generator "works." It is interesting to note that the period of this generator may well be much longer than that of the underlying flat random number generator.

It is easy to see as a general rule that when we try to optimize a library routine for minimum mean running time (as against the Chebyshev minimax run time) we need to consider the distribution of the input data. Hence floating point numerical routines need to consider the reciprocal distribution; the square root, log, exponential, and sine

* Thanks to Brian Kernighan.

TABLE V—DISTRIBUTION OF 8192 RANDOM MANTISSAS

N	χ^2	Degrees of Freedom	Residuals	
			Sign Changes	Expected
64	61.392	63	30	32
32	22.804	31	14	16
16	11.150	15	8	8
8	7.724	7	5	4
4	3.261	3	2	2
2	1.467	1	1	1

are all examples. In the case of the exponential and sine, some study of the exponents is also necessary.

REFERENCES

1. *Handbook of Mathematical Functions*, AMS 55, Nat. Bureau of Standards, 1964.
2. Pinkham, R. S., "On the Distribution of First Significant Digits," Annal. Math. Statistics, *32* (1961), pp. 1223–1230.
3. Adhikari, A. K., and Sarkar, B. P., "Distribution of most Significant Digit in Certain Functions Whose Arguments are Random Variables," Indian J. of Statistics, Series B, *30*, Parts 1 and 2 (1968), pp. 47–58.
4. Raimi, R. A., "On the Distribution of First Significant Digits," Amer. Math. Monthly, *74*, No.2 (February 1969), pp. 342–348.
5. Hamming, R. W., *Numerical Methods for Scientists and Engineers*, New York: McGraw-Hill, 1962.
6. Hamming, R. W., and Mammel, W. L., "A Note on the Location of the Binary Point in a Computing Machine," IEEE Trans. Electronic Computers, *EC-14*, No. 2 (February 1965), pp. 260–1.
7. McKeeman, W. M., "Representation Error for Real Number in Binary Computer Arithmetic," IEEE Trans. Electronic Computers, *EC-16*, No. 6 (June 1967), pp. 682.